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Some new bounds on the spectral radius of matrices ☆

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ABSTRACT

A new lower bound on the smallest eigenvalue $\tau(A \star B)$ for the Fan product of two nonsingular M-matrices A and B is given. Meanwhile, we also obtain a new upper bound on the spectral radius $\rho(A \circ B)$ for nonnegative matrices A and B. These bounds improve some results of Huang (2008) [R. Huang, Some inequalities for the Hadamard product and the Fan product of matrices, Linear Algebra Appl. 428 (2008) 1551–1559].

1. Introduction

For a positive integer n, N denotes the set $\{1, 2, \ldots, n\}$. The set of all $n \times n$ complex matrices is denoted by $C^{n\times n}$ and $R^{n\times n}$ denotes the set of all $n \times n$ real matrices throughout.

Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be two real $n \times n$ matrices. Then, $A \geqslant B (> B)$ if $a_{i,j} \geqslant b_{i,j} (> b_{i,j})$ for all $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n$. If O is the null matrix and $A \geqslant O (> O)$, we say that A is a nonnegative (positive)

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matrix. The spectral radius of A is denoted by $\rho(A)$. If A is a nonnegative matrix, the Perron–Frobenius theorem guarantees that $\rho(A) \in \sigma(A)$, where $\sigma(A)$ denotes the spectrum of A.

For $n \geqslant 2$, an $n \times n$ $A \in C^{n\times n}$ is reducible if there exists an $n \times n$ permutation matrix P such that

$$P^TAP = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix},$$

where $A_{1,1}$ is an $r \times r$ submatrix and $A_{2,2}$ is an $(n-r) \times (n-r)$ submatrix, where $1 \leqslant r < n$. If no such permutation matrix exists, then A is irreducible. If A is a 1×1 complex matrix, then A is irreducible if its single entry is nonzero, and reducible otherwise.

Let A be an irreducible nonnegative matrix. It is well known that there exists a positive vector u such that $Au = \rho(A)u$, u being called right Perron eigenvector of A.

The *Hadamard product* of $A \in C^{n\times n}$ and $B \in C^{n\times n}$ is defined by $A \circ B \equiv (a_{i,j}b_{i,j}) \in C^{n\times n}$.

In [3, p. 358], there is a simple estimate for $\rho(A\circ B)$: if $A, B \in R^{n\times n}$, $A \geqslant O$, and $B \geqslant O$, then $\rho(A \circ B) \leqslant \rho(A)\rho(B)$. From Exercise [3, p. 358], we know this inequality can be very weak by taking $B = J$, the matrix of all ones. For example, If $A = I$, $B = J$, then we have

$$\rho(A\circ B) = \rho(A) = 1 \ll \rho(A)\rho(B) = n$$

when n is very large. But also clearly show that equality can occur(let $A = I$ and $B = I$).

For a nonnegative matrix $A = (a_{i,j})$, let $Z(A) = D(A) - A$, where $D(A) = diag(a_{i,i})$. We denote $J'_A = -D_1^{-1}Z(A)$ with $D_1 = diag(d_{i,i})$ and the same expression for J'_B, where

$$d_{i,i} = \begin{cases} a_{i,i}, & \textit{if } a_{i,i} \neq 0, \\ 1, & \textit{if } a_{i,i} = 0. \end{cases}$$

For two nonnegative matrices A and B, recently, Huang [4] gave some upper bounds for $\rho(A \circ B)$, that is,

(1) If $a_{i,i}b_{i,i} \neq 0$ for all i, then

$$\rho(A\circ B) \leqslant (1 + \rho(J'_A)\rho(J'_B)) \max_{1\leqslant i\leqslant n} a_{i,i}b_{i,i}. \tag{1}$$

(2) If $a_{i_0i_0} \neq 0$ or $b_{i_0i_0} \neq 0$ for some i_0, but $a_{i,i}b_{i,i} = 0$ for all i, then

$$\rho(A\circ B) \leqslant \rho(J'_A)\rho(J'_B) \max_{1\leqslant i\leqslant n} \{a_{i,i}, b_{i,i}\}. \tag{2}$$

(3) If $a_{i,i} = 0$ and $b_{i,i} = 0$ for all i, then

$$\rho(A\circ B) \leqslant \rho(J'_A)\rho(J'_B). \tag{3}$$

(4) If $a_{i_0i_0}b_{i_0i_0} \neq 0$ and $a_{j_0j_0}b_{j_0j_0} \neq 0$ for some i_0 and j_0, then the upper bound on $\rho(A \circ B)$ is the maximum value of the upper bounds of (1)–(3).

The set $Z_n \subset R^{n\times n}$ is defined by

$$Z_n = \{A = (a_{i,j}) \in R^{n\times n} : a_{i,j} \leqslant 0 \quad \textit{if } i \neq j,\ i,j = 1,\ldots,n\}.$$

If $A = (a_{i,j}) \in Z_n$, and if we denote $\min\{Re(\lambda) : \lambda \in \sigma(A)\}$ by $\tau(A)$. Basic for our purpose are the following simple facts (see Problems 16, 19 and 28 in Section 2.5 of [3]):

(i) $\tau(A) \in \sigma(A)$; $\tau(A)$ is called the minimum eigenvalue of A.
(ii) If $A, B \in Z_n$, and $A \geqslant B$, then $\tau(A) \geqslant \tau(B)$.
(iii) If $A \in Z_n$, then $\rho(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix A^{-1}, and $\tau(A) = \frac{1}{\rho(A^{-1})}$ is a positive real eigenvalue of A.

Let A be an irreducible nonsingular M-matrix. It is well known that there exists a positive vector u such that $Au = \tau(A)u$, u being called right Perron eigenvector of A.

Let $A \in C^{n\times n}, B \in C^{n\times n}$. The *Fan product* of A and B is denoted by $A \star B \equiv C = (c_{i,j}) \in C^{n\times n}$ and is defined by

$$c_{i,j} = \begin{cases} -a_{i,j}b_{i,j}, & \textit{if } i \neq j, \\ a_{i,i}b_{i,i}, & \textit{if } i = j. \end{cases}$$

If $A, B \in Z_n$ are M-matrices, then so is $A \star B$. In [3, p. 359], a lower bound for $\tau(A \star B)$ was given: Let $A, B \in Z_n$ be M-matrices. Then $A^{-1} \circ B^{-1} \geqslant (A \star B)^{-1}$, and hence $\tau(A \star B) \geqslant \tau(A)\tau(B)$.

If $A = (a_{i,j})$ is a nonsingular M-matrix, Let $Z(A) = A - D(A)$, where $D(A) = diag(a_{i,i})$. It is clear that $a_{i,i} > 0$. We can define $J_A = -D^{-1}Z(A)$, then J_A is nonnegative. Huang [4] gave a sharper lower bound for $\tau(A \star B)$, that is,

$$\tau(A \star B) \geqslant (1 - \rho(J_A)\rho(J_B)) \min_{1 \leqslant i \leqslant n} (a_{i,i}b_{i,i}). \tag{4}$$

In this paper, we first give an upper bound on $\rho(J_A)$ for an M-matrix A in Section 2. Meanwhile, we give an upper bound on $\rho(A \circ B)$ for two nonnegative matrices A and B in Section 3 and a lower bound on $\tau(A \star B)$ for two M-matrices A and B in Section 4, some examples are given to illustrate our results.

2. An upper bound for the spectral radius of J_A

Lemma 1 [1]. *Let $A \in R^{n\times n}$ be a nonnegative matrix. If A_k is a principal submatrix of A, then $\rho(A_k) \leqslant \rho(A)$. If, in addition, A is irreducible and $A_k \neq A$, then $\rho(A_k) < \rho(A)$.*

Lemma 2 [2]. *Let $A = (a_{i,j}) \in R^{n\times n}$ be a nonnegative matrix. Then*

$$\rho(A) \leqslant \max_{i\neq j} \frac{1}{2}\left\{a_{i,i} + a_{j,j} + \left[(a_{i,i} - a_{j,j})^2 + 4\sum_{k\neq i} a_{i,k} \sum_{k\neq j} a_{j,k}\right]^{\frac{1}{2}}\right\}.$$

In addition, if A is irreducible, then

$$\rho(A) \geqslant \min_{i\neq j} \frac{1}{2}\left\{a_{i,i} + a_{j,j} - \left[(a_{i,i} - a_{j,j})^2 + 4\sum_{k\neq i} a_{i,k} \sum_{k\neq j} a_{j,k}\right]^{\frac{1}{2}}\right\}.$$

Theorem 1. *Let $A = (a_{i,j})$ is a nonsingular M-matrix. Then*

$$\rho(J_A) \leqslant \max_{i\neq j} \sqrt{\left(1 - \frac{\tau(A)}{a_{i,i}}\right)\left(1 - \frac{\tau(A)}{a_{j,j}}\right)}.$$

In addition, if A is irreducible, then

$$\rho(J_A) \geqslant \min_{i\neq j} \sqrt{\left(1 - \frac{\tau(A)}{a_{i,i}}\right)\left(1 - \frac{\tau(A)}{a_{j,j}}\right)}.$$

Proof. Let $A = (a_{i,j})$ be a nonsingular M-matrix. If A is irreducible, then there exists a positive vector w such that $Aw = \tau(A)w$ and $a_{i,i} - \tau(A) > 0, \forall i \in N$. Thus, we have

$$\sum_{j\neq i} \frac{|a_{i,j}|w_j}{a_{i,i}w_i} = 1 - \frac{\tau(A)}{a_{i,i}}. \quad \square \tag{5}$$

Define $W = diag(w_1, \ldots, w_n)$, it is clear that W is a positive diagonal matrix. Let $\widetilde{J_A} = W^{-1}J_AW = (t_{i,j})$, then we have

$$\widetilde{J_A} = W^{-1}J_AW = (t_{i,j}) = \begin{bmatrix} 0 & \frac{a_{1,2}w_2}{a_{1,1}w_1} & \cdots & \frac{a_{1,n}w_n}{a_{1,1}w_1} \\ \frac{a_{2,1}w_1}{a_{2,2}w_2} & 0 & \cdots & \frac{a_{2,n}w_n}{a_{2,2}w_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n,1}w_1}{a_{n,n}w_n} & \frac{a_{n,2}w_2}{a_{n,n}w_n} & \cdots & 0 \end{bmatrix}.$$

Since J_A is nonnegative irreducible, we know that $\widetilde{J_A}$ is also nonnegative irreducible and $\rho(\widetilde{J_A}) = \rho(J_A)$. For $\widetilde{J_A}$, from Lemma 2, we have

$$\begin{aligned} \rho(\widetilde{J_A}) &\leqslant \max_{i\neq j} \frac{1}{2}\left\{t_{i,i} + t_{j,j} + \left[(t_{i,i} - t_{j,j})^2 + 4\sum_{k\neq i} t_{i,k}\sum_{k\neq j} t_{j,k}\right]^{\frac{1}{2}}\right\} \\ &= \max_{i\neq j} \frac{1}{2}\left[4\sum_{k\neq i} t_{i,k}\sum_{k\neq j} t_{j,k}\right]^{\frac{1}{2}}. \end{aligned} \tag{6}$$

From (5), we have

$$\sum_{k\neq i} t_{i,k} = 1 - \frac{\tau(A)}{a_{i,i}}, \quad \sum_{k\neq j} t_{j,k} = 1 - \frac{\tau(A)}{a_{j,j}}. \tag{7}$$

Thus, from (6) and (7), we have

$$\rho(J_A) = \rho(\widetilde{J_A}) \leqslant \max_{i\neq j}\sqrt{\left(1 - \frac{\tau(A)}{a_{i,i}}\right)\left(1 - \frac{\tau(A)}{a_{j,j}}\right)}.$$

Similarly, from Lemma 2, we have

$$\rho(J_A) = \rho(\widetilde{J_A}) \geqslant \min_{i\neq j}\sqrt{\left(1 - \frac{\tau(A)}{a_{i,i}}\right)\left(1 - \frac{\tau(A)}{a_{j,j}}\right)}.$$

If A is reducible. It is well known that a matrix in Z_n is a nonsingular M-matrix if and only if all its leading principal minors are positive (see condition (E17) of Theorem 6.2.3 of [1]). If we denote by $D = (d_{i,j})$ the $n \times n$ permutation matrix with $d_{1,2} = d_{2,3} = \cdots = d_{n-1,n} = d_{n,1} = 1$, the remaining $d_{i,j}$ zero, then both $A - tD$ and $B - tD$ are irreducible nonsingular M-matrices for any chosen positive real number t, sufficiently small such that all the leading principal minors of both $A - tD$ and $B - tD$ are positive. Now we substitute $A - tD$ for A in the previous case, and then letting $t \to 0$, the result follows by continuity.

Remark 1. Since $1 - \frac{\tau(A)}{a_{i,i}} > 0$ and $1 - \frac{\tau(A)}{a_{j,j}} > 0$, then we know that

$$\sqrt{\left(1 - \frac{\tau(A)}{a_{i,i}}\right)\left(1 - \frac{\tau(A)}{a_{j,j}}\right)} \leqslant \frac{1}{2}\left(\left(1 - \frac{\tau(A)}{a_{i,i}}\right) + \left(1 - \frac{\tau(A)}{a_{j,j}}\right)\right).$$

Thus, we have

$$\begin{aligned} \max_{i\neq j}\sqrt{\left(1 - \frac{\tau(A)}{a_{i,i}}\right)\left(1 - \frac{\tau(A)}{a_{j,j}}\right)} &\leqslant \frac{1}{2}\max_{i\neq j}\left(\left(1 - \frac{\tau(A)}{a_{i,i}}\right) + \left(1 - \frac{\tau(A)}{a_{j,j}}\right)\right) \\ &\leqslant \frac{1}{2}\left(\max_{1\leqslant i\leqslant n}\left(1 - \frac{\tau(A)}{a_{i,i}}\right) + \max_{1\leqslant j\leqslant n}\left(1 - \frac{\tau(A)}{a_{j,j}}\right)\right) \\ &= \max_{1\leqslant i\leqslant n}\left(1 - \frac{\tau(A)}{a_{i,i}}\right) \leqslant 1 - \frac{\tau(A)}{\max_{1\leqslant i\leqslant n} a_{i,i}}. \end{aligned}$$

If A is irreducible, then we have

$$\min_{i\neq j}\sqrt{\left(1-\frac{\tau(A)}{a_{i,i}}\right)\left(1-\frac{\tau(A)}{a_{j,j}}\right)} \geqslant \sqrt{\min_{1\leqslant i\leqslant n}\left(1-\frac{\tau(A)}{a_{i,i}}\right)\min_{1\leqslant j\leqslant n}\left(1-\frac{\tau(A)}{a_{j,j}}\right)}$$
$$=\min_{1\leqslant i\leqslant n}\left(1-\frac{\tau(A)}{a_{i,i}}\right)\geqslant 1-\frac{\tau(A)}{\min\limits_{1\leqslant i\leqslant n} a_{i,i}}.$$

Example 1 [4]. Let

$$A=\begin{bmatrix}4 & -3\\ -1 & 2\end{bmatrix}. \tag{8}$$

Then $\tau(A)=1$, by Theorem 1 of [4], we have

$$1-\frac{\tau(A)}{\min\limits_{1\leqslant i\leqslant 2} a_{i,i}}=\frac{1}{2}<\rho(J_A)=\frac{\sqrt{6}}{4}<\frac{3}{4}=1-\frac{\tau(A)}{\max\limits_{1\leqslant i\leqslant 2} a_{i,i}}.$$

By Theorem 1 in this paper, we have

$$\frac{\sqrt{6}}{4}=\min_{i\neq j}\sqrt{\left(1-\frac{\tau(A)}{a_{i,i}}\right)\left(1-\frac{\tau(A)}{a_{j,j}}\right)}\leqslant\rho(J_A)$$
$$\leqslant\max_{i\neq j}\sqrt{\left(1-\frac{\tau(A)}{a_{i,i}}\right)\left(1-\frac{\tau(A)}{a_{j,j}}\right)}=\frac{\sqrt{6}}{4}.$$

3. A lower bound for the minimum eigenvalue of the Fan product of M-matrices

In this Section, we will give a lower bound for $\tau(A\star B)$.

Lemma 3. *Let A, B be two nonsingular M-matrices and if D and E are two positive diagonal matrices, then*

$$D(A\star B)E=(DAE)\star B=(DA)\star(BE)=(AE)\star(DB)=A\star(DBE).$$

Proof. Lemma 3 follows from Definition of *Fan product*. $\square$

Theorem 2. *Let $A, B\in R^{n\times n}$ be two nonsingular M-matrices. Then*

$$\tau(A\star B)\geqslant\min_{i\neq j}\frac{1}{2}\left\{a_{i,i}b_{i,i}+a_{j,j}b_{j,j}-\left[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2+4a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho^2(J_A)\rho^2(J_B)\right]^{\frac{1}{2}}\right\}. \tag{9}$$

Proof. It is quite evident that (13) holds with equality for $n=1$. $\square$

We next assume that $n\geqslant 2$.

If $A\star B$ is irreducible, then A and B are irreducible. Obviously J_A and J_B are also irreducible and nonnegative. Then there exists two positive vectors $u=(u_i)$ and $v=(v_i)$ such that $J_Au=\rho(J_A)u$ and $J_Bv=\rho(J_B)v$, thus, we have

$$\sum_{j\neq i}\frac{|a_{i,j}|u_j}{u_i}=a_{i,i}\rho(J_A) \tag{10}$$

and

$$\sum_{j\neq i}\frac{|b_{i,j}|v_j}{v_i}=b_{i,i}\rho(J_B). \tag{11}$$

Let $\widehat{A}=(\hat{a}_{i,j})=\widehat{U}^{-1}A\widehat{U}$ and $\widehat{B}=(\hat{b}_{i,j})=\widehat{V}^{-1}B\widehat{V}$ in which $\widehat{U}$ and $\widehat{V}$ are the nonsingular diagonal matrices $\widehat{U}=diag(u_1,\ldots,u_n)$ and $\widehat{V}=diag(v_1,\ldots,v_n)$. Then, we have

$$\widehat{A}=(\hat{a}_{i,j})=\widehat{U}^{-1}A\widehat{U}=\begin{bmatrix} a_{1,1} & \frac{a_{1,2}u_2}{u_1} & \cdots & \frac{a_{1,n}u_n}{u_1}\\ \frac{a_{2,1}u_1}{u_2} & a_{2,2} & \cdots & \frac{a_{2,n}u_n}{u_2}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{a_{n,1}u_1}{u_n} & \frac{a_{n,2}u_2}{u_n} & \cdots & a_{n,n}\end{bmatrix},$$

$$\widehat{B}=(\hat{b}_{i,j})=\widehat{V}^{-1}B\widehat{V}=\begin{bmatrix} b_{1,1} & \frac{b_{1,2}v_2}{v_1} & \cdots & \frac{b_{1,n}v_n}{v_1}\\ \frac{b_{2,1}v_1}{v_2} & b_{2,2} & \cdots & \frac{b_{2,n}v_n}{v_2}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{b_{n,1}v_1}{v_n} & \frac{b_{n,2}v_2}{v_n} & \cdots & b_{n,n}\end{bmatrix}.$$

It is easy to see that $\widehat{A}$ and $\widehat{B}$ are irreducible M-matrices since A and B are.

Also let $\widehat{W}=\widehat{V}\widehat{U}$, then $\widehat{W}$ is nonsingular. From Lemma 3, we have

$$\begin{aligned}(\widehat{V}\widehat{U})^{-1}(A\star B)(\widehat{V}\widehat{U}) &= \widehat{U}^{-1}\widehat{V}^{-1}(A\star B)\widehat{V}\widehat{U}\\ &= \widehat{U}^{-1}(A\star(\widehat{V}^{-1}B\widehat{V}))\widehat{U}\\ &= (\widehat{U}^{-1}A\widehat{U})\star(\widehat{V}^{-1}B\widehat{V})\\ &= \widehat{A}\star\widehat{B}.\end{aligned}$$

Thus, we have $\tau(\widehat{A}\star\widehat{B})=\tau(A\star B)$ and

$$\widehat{A}\star\widehat{B}=(s_{i,j})=\begin{bmatrix} a_{1,1}b_{1,1} & -\frac{a_{1,2}b_{1,2}u_2v_2}{u_1v_1} & \cdots & -\frac{a_{1,n}b_{1,n}u_nv_n}{u_1v_1}\\ -\frac{a_{2,1}b_{2,1}u_1v_1}{u_2v_2} & a_{2,2}b_{2,2} & \cdots & -\frac{a_{2,n}b_{2,n}u_nv_n}{u_2v_2}\\ \vdots & \vdots & \ddots & \vdots\\ -\frac{a_{n,1}b_{n,1}u_1v_1}{u_nv_n} & -\frac{a_{n,2}b_{n,2}u_2v_2}{u_nv_n} & \cdots & a_{n,n}b_{n,n}\end{bmatrix}. \tag{12}$$

We next consider the minimum eigenvalue $\tau(\widehat{A}\star\widehat{B})$ of $\widehat{A}\star\widehat{B}$. For the irreducible nonsingular M-matrices $\widehat{A},\widehat{B}$, let $\tau(\widehat{A}\star\widehat{B})=\lambda$, so that $0<\lambda<a_{i,i}b_{i,i},\forall i\in N$. From the definition of the *Fan product* of $\widehat{A}$ and $\widehat{B}$, (10)-(12) and Theorem 1.23 of [5], there is a pair (i,j) of positive integers with $i\neq j$ such that

$$|\lambda-a_{i,i}b_{i,i}||\lambda-a_{j,j}b_{j,j}|\leqslant\sum_{k\neq i}|s_{i,k}|\sum_{k\neq j}|s_{j,k}|.$$

Observe that

$$\begin{aligned}\sum_{k\neq i}|s_{i,k}|\sum_{k\neq j}|s_{j,k}| &= \sum_{k\neq i}\left|-\frac{a_{i,k}b_{i,k}u_kv_k}{u_iv_i}\right|\sum_{k\neq j}\left|-\frac{a_{j,k}b_{j,k}u_kv_k}{u_jv_j}\right|\\ &\leqslant \sum_{k\neq i}\frac{|a_{i,k}|u_k}{u_i}\sum_{k\neq i}\frac{|b_{i,k}|v_k}{v_i}\sum_{k\neq j}\frac{|a_{j,k}|u_k}{u_j}\sum_{k\neq j}\frac{|b_{j,k}|v_k}{v_j}\\ &= a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho^2(J_A)\rho^2(J_B).\end{aligned}$$

Thus, we have

$$|\lambda - a_{i,i}b_{i,i}||\lambda - a_{j,j}b_{j,j}| \leqslant a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho^2(J_A)\rho^2(J_B). \tag{13}$$

From inequality (13) and $0 < \lambda < a_{i,i}b_{i,i}, \forall i \in N$, we have

$$(\lambda - a_{i,i}b_{i,i})(\lambda - a_{j,j}b_{j,j}) \leqslant a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho^2(J_A)\rho^2(J_B). \tag{14}$$

Thus, from inequality (14), we have

$$\lambda \geqslant \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 + 4a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho^2(J_A)\rho^2(J_B)\right]^{\frac{1}{2}}\right\}.$$

That is

$$\begin{aligned}\tau(A\star B) &\geqslant \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 + 4a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho^2(J_A)\rho^2(J_B)\right]^{\frac{1}{2}}\right\}\\ &\geqslant \min_{i\neq j}\frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 + 4a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho^2(J_A)\rho^2(J_B)\right]^{\frac{1}{2}}\right\}.\end{aligned}$$

Now, assume that $A\star B$ is reducible. It is well known that a matrix in Z_n is a nonsingular M-matrix if and only if all its leading principal minors are positive(see condition (E17) of Theorem 6.2.3 of [1]). If we denote by $D = (d_{i,j})$ the $n \times n$ permutation matrix with $d_{1,2} = d_{2,3} = \cdots = d_{n-1,n} = d_{n,1} = 1$, the remaining $d_{i,j}$ zero, then both $A - tD$ and $B - tD$ are irreducible nonsingular M-matrices for any chosen positive real number t, sufficiently small such that all the leading principal minors of both $A - tD$ and $B - tD$ are positive. Now we substitute $A - tD$ and $B - tD$ for A and B, respectively in the previous case, and then letting $t \to 0$, the result follows by continuity.

Remark 2. We next give a simple comparison between the lower bound in (9) and the lower bound in [4]. Without loss of generality, for $i \neq j$, assume that

$$(1 - \rho(J_A)\rho(J_B))a_{i,i}b_{i,i} \leqslant (1 - \rho(J_A)\rho(J_B))a_{j,j}b_{j,j}. \tag{15}$$

Thus, we can write (15) equivalently as

$$a_{j,j}b_{j,j}\rho(J_A)\rho(J_B) \leqslant a_{i,i}b_{i,i}\rho(J_A)\rho(J_B) - a_{i,i}b_{i,i} + a_{j,j}b_{j,j}. \tag{16}$$

From (9) and (16), we have

$$\begin{aligned}&\frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 + 4a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho^2(J_A)\rho^2(J_B)\right]^{\frac{1}{2}}\right\}\\ &\geqslant \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2\right.\right.\\ &\left.\left.+ 4a_{i,i}b_{i,i}\rho(J_A)\rho(J_B)[a_{i,i}b_{i,i}\rho(J_A)\rho(J_B) - (a_{i,i}b_{i,i} - a_{j,j}b_{j,j})]\right]^{\frac{1}{2}}\right\}\\ &= \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2\right.\right.\\ &\left.\left.+ 4a_{i,i}^2b_{i,i}^2\rho(J_A)^2\rho(J_B)^2 - 4a_{i,i}b_{i,i}(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})\rho(J_A)\rho(J_B)\right]^{\frac{1}{2}}\right\}\\ &= \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j} - 2a_{i,i}b_{i,i}\rho(J_A)\rho(J_B))^2\right]^{\frac{1}{2}}\right\}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - (a_{j,j}b_{j,j} - a_{i,i}b_{i,i} + 2a_{i,i}b_{i,i}\rho(J_A)\rho(J_B))\right\} \\
&= (1 - \rho(J_A)\rho(J_B))a_{i,i}b_{i,i}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\tau(A \star B) &\geqslant \min_{i \neq j} \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 + 4a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho^2(J_A)\rho^2(J_B)\right]^{\frac{1}{2}}\right\} \\
&\geqslant (1 - \rho(J_A)\rho(J_B)) \min_{1 \leqslant i \leqslant n}(a_{i,i}b_{i,i}).
\end{aligned}
$$

From Theorems 1, 2 and Remark 1 we can obtain the following corollary:

Corollary 1. *Let $A = (a_{i,j}), B = (b_{i,j})$ be two nonsingular M-matrices. Then we have*

$$
\begin{aligned}
\tau(A \star B) &\geqslant \min_{i \neq j} \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 + 4a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho^2(J_A)\rho^2(J_B)\right]^{\frac{1}{2}}\right\} \\
&\geqslant \min_{i \neq j} \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2\right.\right. \\
&\quad + 4\left(a_{i,i} - \frac{a_{i,i}\tau(A)}{\max_{1 \leqslant i \leqslant n} a_{i,i}}\right)\left(b_{i,i} - \frac{b_{i,i}\tau(B)}{\max_{1 \leqslant i \leqslant n} b_{i,i}}\right) \\
&\quad \left.\left. \times \left(a_{j,j} - \frac{a_{j,j}\tau(A)}{\max_{1 \leqslant i \leqslant n} a_{i,i}}\right)\left(b_{j,j} - \frac{b_{j,j}\tau(B)}{\max_{1 \leqslant i \leqslant n} b_{i,i}}\right)\right]^{\frac{1}{2}}\right\}.
\end{aligned}
$$

From Theorem 2, Remark 2 and [1, p. 380] we can obtain the following corollary:

Corollary 2. *Let $A = (a_{i,j}), B = (b_{i,j})$ be two nonsingular M-matrices. Then we have*

$$
\begin{aligned}
|det(A \star B)| &\geqslant [\tau(A \star B)]^n \\
&\geqslant \min_{i \neq j} \frac{1}{2^n}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 + 4a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho^2(J_A)\rho^2(J_B)\right]^{\frac{1}{2}}\right\}^n \\
&\geqslant (1 - \rho(J_A)\rho(J_B))^n \min_{1 \leqslant i \leqslant n}(a_{i,i}b_{i,i})^n.
\end{aligned}
$$

Example 2. Let

$$
A = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -0.5 \\ -0.5 & 1 \end{bmatrix}.
$$

Then

$$
A \star B = \begin{bmatrix} 6 & -1 \\ -0.5 & 2 \end{bmatrix}.
$$

It is easy to show that $\rho(J_A) = 0.5774, \rho(J_B) = 0.3536$ and $\tau(A \star B) = 1.8787$. By Corollary 5.7.4.1 in [3], we have that $\tau(A \star B) \geqslant \tau(A)\tau(B) = 1 \times 0.7929 = 0.7929$. By Theorem 4 in [4], we have that $\tau(A \star B) \geqslant (1 - \rho(J_A)\rho(J_B)) \min_{1 \leqslant i \leqslant 2}(a_{i,i}b_{i,i}) = 1.5918$. By Theorem 2 in this paper, we have that

$$
\begin{aligned}
\tau(A \star B) &\geqslant \min_{i \neq j} \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} - \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2\right.\right. \\
&\quad \left.\left. + 4a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho^2(J_A)\rho^2(J_B)\right]^{\frac{1}{2}}\right\} = 1.8787.
\end{aligned}
$$

4. An upper bound for the Hadamard product of nonnegative matrices

In this section, for $A, B \geqslant 0$, we will give an upper bound for $\rho(A \circ B)$. Similar to [4], for $A = (a_{i,j}) \geqslant 0$, write $N = A - D$, where $D = diag(a_{i,i})$. We denote $J_A' = D_1 N$ with $D_1 = diag(d_{i,i})$, where

$$d_{i,i} = \begin{cases} a_{i,i}, & \text{if } a_{i,i} \neq 0, \\ 1, & \text{if } a_{i,i} = 0. \end{cases}$$

Note that J_A' is nonnegative, and $J_A' = A$ if $a_{i,i} = 0$ for all i. For $B = (b_{i,j}) \geqslant 0$, let $D_2 = diag(s_{i,i})$, where

$$s_{i,i} = \begin{cases} b_{i,i}, & \text{if } b_{i,i} \neq 0, \\ 1, & \text{if } b_{i,i} = 0. \end{cases}$$

Similarly, the nonnegative matrix J_B' is defined.

Lemma 4 [1]. *Let $A \in R^{n\times n}$ be given. Then either A is irreducible or there exists a permutation P such that*

$$P^TAP = \begin{bmatrix} R_{1,1} & R_{1,2} & \cdots & R_{1,m} \\ O & R_{2,2} & \cdots & R_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & R_{m,m} \end{bmatrix}, \tag{17}$$

where each square submatrix $R_{j,j}$, $1 \leqslant j \leqslant m$, is either irreducible or a 1×1 null matrix.

Remark 3. Eq. (17) is said to be the normal form of a matrix A. Clearly, the eigenvalues of A are the eigenvalues of the square submatrices $R_{j,j}$, $1 \leqslant j \leqslant m$ (cf. [5]).

Lemma 5 [3]. *Let $A, B \in C^{n\times n}$ and if $D \in C^{n\times n}$ and $E \in C^{n\times n}$ are diagonal, then*

$$D(A \circ B)E = (DAE) \circ B = (DA) \circ (BE) = (AE) \circ (DB) = A \circ (DBE).$$

Theorem 3. *Let $A, B \in R^{n\times n}$, $A \geqslant O$, and $B \geqslant O$. Then*

(1) *If $a_{i,i}b_{i,i} \neq 0$ for all i, then*

$$\rho(A \circ B) \leqslant \max_{i\neq j} \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} + \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 + 4a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho^2(J_A')\rho^2(J_B')\right]^{\frac{1}{2}}\right\}. \tag{18}$$

(2) *If $a_{i_0i_0} \neq 0$ and $a_{j_0j_0} \neq 0$ or $b_{i_0i_0} \neq 0$ and $b_{j_0j_0} \neq 0$ for some i_0, j_0 but $a_{i,i}b_{i,i} = 0$ for all i, then*

$$\rho(A \circ B) \leqslant \rho(J_A')\rho(J_B') \max_{1\leqslant i\neq j\leqslant n} \left\{\sqrt{a_{i,i}a_{j,j}}, \sqrt{b_{i,i}b_{j,j}}\right\}. \tag{19}$$

(3) *If $a_{i,i} = 0$ and $b_{i,i} = 0$ for all i, then*

$$\rho(A \circ B) \leqslant \rho(J_A')\rho(J_B'). \tag{20}$$

(4) *If $a_{i_0i_0}b_{i_0i_0} \neq 0$ and $a_{j_0j_0}b_{j_0j_0} \neq 0$ for some i_0 and j_0, then the upper bound on $\rho(A \circ B)$ is the maximum value of the upper bounds of* (18)–(20).

Proof. It is clear that (4) holds with equality for $n = 1$.
We next assume that $n \geqslant 2$. □

If $A \circ B$ is irreducible, then A and B are irreducible. Obviously J_A' and J_B' are also irreducible and nonnegative. Then there exists two positive vectors u, v such that $J_A'u = \rho(J_A')u, J_B'v = \rho(J_B')v$. Thus, we have

$$\sum_{j\neq i}\frac{a_{i,j}u_j}{u_i}=d_{i,i}\rho(J'_A) \tag{21}$$

and

$$\sum_{j\neq i}\frac{b_{i,j}v_j}{v_i}=s_{i,i}\rho(J'_B). \tag{22}$$

Let $\widetilde{A}=(\tilde{a}_{i,j})=\widetilde{U}^{-1}A\widetilde{U}$ and $\widetilde{B}=(\tilde{b}_{i,j})=\widetilde{V}^{-1}B\widetilde{V}$ in which $\widetilde{U}$ and $\widetilde{V}$ are nonsingular diagonal matrices $\widetilde{U}=diag(u_1,\ldots,u_n)$ and $\widetilde{V}=diag(v_1,\cdots,v_n)$. Then, we have

$$\widetilde{A}=(\tilde{a}_{i,j})=\widetilde{U}^{-1}A\widetilde{U}=\begin{bmatrix} a_{1,1} & \frac{a_{1,2}u_2}{u_1} & \cdots & \frac{a_{1,n}u_n}{u_1}\\ \frac{a_{2,1}u_1}{u_2} & a_{2,2} & \cdots & \frac{a_{2,n}u_n}{u_2}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{a_{n,1}u_1}{u_n} & \frac{a_{n,2}u_2}{u_n} & \cdots & a_{n,n}\end{bmatrix},$$

$$\widetilde{B}=(\tilde{b}_{i,j})=\widetilde{V}^{-1}B\widetilde{V}=\begin{bmatrix} b_{1,1} & \frac{b_{1,2}v_2}{v_1} & \cdots & \frac{b_{1,n}v_n}{v_1}\\ \frac{b_{2,1}v_1}{v_2} & b_{2,2} & \cdots & \frac{b_{2,n}v_n}{v_2}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{b_{n,1}v_1}{v_n} & \frac{b_{n,2}v_2}{v_n} & \cdots & b_{n,n}\end{bmatrix}.$$

It is easy to see that $\widetilde{A}$ and $\widetilde{B}$ are nonnegative irreducible matrices since A and B are.

Also let $\widetilde{W}=\widetilde{V}\widetilde{U}$, then $\widetilde{W}$ is nonsingular. From Lemma 5, we have

$$\begin{aligned}(\widetilde{V}\widetilde{U})^{-1}(A\circ B)(\widetilde{V}\widetilde{U}) &= \widetilde{U}^{-1}\widetilde{V}^{-1}(A\circ B)\widetilde{V}\widetilde{U}\\ &= \widetilde{U}^{-1}(A\circ(\widetilde{V}^{-1}B\widetilde{V}))\widetilde{U}\\ &= (\widetilde{U}^{-1}A\widetilde{U})\circ(\widetilde{V}^{-1}B\widetilde{V})\\ &= \widetilde{A}\circ\widetilde{B}.\end{aligned}$$

Thus, we have that $\rho(A\circ B)=\rho(\widetilde{A}\circ\widetilde{B})$ and

$$\widetilde{A}\circ\widetilde{B}=(t_{i,j})=\begin{bmatrix} a_{1,1}b_{1,1} & \frac{a_{1,2}b_{1,2}u_2v_2}{u_1v_1} & \cdots & \frac{a_{1,n}b_{1,n}u_nv_n}{u_1v_1}\\ \frac{a_{2,1}b_{2,1}u_1v_1}{u_2v_2} & a_{2,2}b_{2,2} & \cdots & \frac{a_{2,n}b_{2,n}u_nv_n}{u_2v_2}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{a_{n,1}b_{n,1}u_1v_1}{u_nv_n} & \frac{a_{n,2}b_{n,2}u_2v_2}{u_nv_n} & \cdots & a_{n,n}b_{n,n}\end{bmatrix}. \tag{23}$$

We next consider the spectral radius $\rho(\widetilde{A}\circ\widetilde{B})$ of $\widetilde{A}\circ\widetilde{B}$. For nonnegative irreducible matrices $\widetilde{A}$ and $\widetilde{B}$, from the definition of the *Hadamard product* of $\widetilde{A}$ and $\widetilde{B}$, (21)–(23) and Lemma 2, we have

$$\begin{aligned}\rho(\widetilde{A}\circ\widetilde{B}) &\leqslant \max_{i\neq j}\frac{1}{2}\left\{\tilde{t}_{i,i}+\tilde{t}_{j,j}+\left[(\tilde{t}_{i,i}-\tilde{t}_{j,j})^2+4\sum_{k\neq i}\tilde{t}_{i,k}\sum_{k\neq j}\tilde{t}_{j,k}\right]^{\frac{1}{2}}\right\}\\ &= \max_{i\neq j}\frac{1}{2}\left\{a_{i,i}b_{i,i}+a_{j,j}b_{j,j}+\left[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2\right.\right.\\ &\quad\left.\left.+4\sum_{k\neq i}\frac{a_{i,k}u_k}{u_i}\frac{b_{i,k}v_k}{v_i}\sum_{k\neq j}\frac{a_{j,k}u_k}{u_j}\frac{b_{j,k}v_k}{v_j}\right]^{\frac{1}{2}}\right\}\end{aligned}$$

$$
\begin{aligned}
&\leqslant \max_{i\neq j} \frac{1}{2}\Bigg\{a_{i,i}b_{i,i}+a_{j,j}b_{j,j}+\Big[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2\\
&\quad +4\left(\sum_{k\neq i}\frac{a_{i,k}u_k}{u_i}\sum_{k\neq i}\frac{b_{i,k}v_k}{v_i}\right)\left(\sum_{k\neq j}\frac{a_{j,k}u_k}{u_j}\sum_{k\neq j}\frac{b_{j,k}v_k}{v_j}\right)\Big]^{\frac{1}{2}}\Bigg\}\\
&= \max_{i\neq j} \frac{1}{2}\left\{a_{i,i}b_{i,i}+a_{j,j}b_{j,j}+\left[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2+\ 4d_{i,i}s_{i,i}d_{j,j}s_{j,j}\rho^2(J_A')\rho^2(J_B')\right]^{\frac{1}{2}}\right\}.
\end{aligned}
\tag{24}
$$

From (24), we have

(1) If $a_{i,i}b_{i,i}\neq 0$ for all i, then

$$\rho(A\circ B)\leqslant \max_{i\neq j}\frac{1}{2}\left\{a_{i,i}b_{i,i}+a_{j,j}b_{j,j}+\left[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2+\ 4a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho^2(J_A')\rho^2(J_B')\right]^{\frac{1}{2}}\right\}.$$

(2) If $a_{i_0i_0}\neq 0$ and $a_{j_0j_0}\neq 0$ or $b_{i_0i_0}\neq 0$ and $b_{j_0j_0}\neq 0$ for some i_0, j_0 but $a_{i,i}b_{i,i}=0$ for all i, then

$$\rho(A\circ B)\leqslant \rho(J_A')\rho(J_B')\max_{1\leqslant i\neq j\leqslant n}\{\sqrt{a_{i,i}a_{j,j}},\sqrt{b_{i,i}b_{j,j}}\}.$$

(3) If $a_{i,i}=0$ and $b_{i,i}=0$ for all i, then

$$\rho(A\circ B)\leqslant \rho(J_A')\rho(J_B').$$

(4) If $a_{i_0i_0}b_{i_0i_0}\neq 0$ and $a_{j_0j_0}b_{j_0j_0}\neq 0$ for some i_0 and j_0, then the upper bound on $\rho(A\circ B)$ is the maximum value of the upper bounds of (18)–(20).

Now, assume that $A\circ B$ is reducible. If we denote by $D=(d_{i,j})$ the $n\times n$ permutation matrix with $d_{1,2}=d_{2,3}=\cdots=d_{n-1,n}=d_{n,1}=1$, the remaining $d_{i,j}$ zero, then both $A+tD$ and $B+tD$ are nonnegative irreducible matrices for any chosen positive real number t. Now we substitute $A+tD$ and $B+tD$ for A and B, respectively in the previous case, and then letting $t\to 0$, the result follows by continuity.

Remark 4. We next give a simple comparison between the upper bound for the case of Theorem 6 in [4] and the upper bound for the case of (1) in Theorem 3 in this paper. Without loss of generality, for $i\neq j$, assume that

$$a_{i,i}b_{i,i}(1+\rho(J_A')\rho(J_B'))\geqslant a_{j,j}b_{j,j}(1+\rho(J_A')\rho(J_B')). \tag{25}$$

Thus, we can write (25) equivalently as

$$a_{j,j}b_{j,j}\rho(J_A')\rho(J_B')\leqslant a_{i,i}b_{i,i}\rho(J_A')\rho(J_B')+a_{i,i}b_{i,i}-a_{j,j}b_{j,j}. \tag{26}$$

From (26), we have

$$
\begin{aligned}
&a_{i,i}b_{i,i}+a_{j,j}b_{j,j}+\left[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2+4a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho^2(J_A')\rho^2(J_B')\right]^{\frac{1}{2}}\\
&\quad\leqslant a_{i,i}b_{i,i}+a_{j,j}b_{j,j}+\Big[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2\\
&\quad +4a_{i,i}b_{i,i}\rho(J_A')\rho(J_B')\left(a_{i,i}b_{i,i}\rho(J_A')\rho(J_B')+a_{i,i}b_{i,i}-a_{j,j}b_{j,j}\right)\Big]^{\frac{1}{2}}\\
&= a_{i,i}b_{i,i}+a_{j,j}b_{j,j}+\Big[(a_{i,i}b_{i,i}-a_{j,j}b_{j,j})^2+4a_{i,i}^2b_{i,i}^2\rho^2(J_A')\rho^2(J_B')
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 4a_{i,i}b_{i,i}\rho(J'_A)\rho(J'_B)(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})\Big]^{\frac{1}{2}} \\
&= a_{i,i}b_{i,i} + a_{j,j}b_{j,j} + \left[\left(a_{i,i}b_{i,i} - a_{j,j}b_{j,j} + 2a_{i,i}b_{i,i}\rho(J'_A)\rho(J'_B)\right)^2\right]^{\frac{1}{2}} \\
&= a_{i,i}b_{i,i} + a_{j,j}b_{j,j} + a_{i,i}b_{i,i} - a_{j,j}b_{j,j} + 2a_{i,i}b_{i,i}\rho(J'_A)\rho(J'_B) \\
&= 2a_{i,i}b_{i,i} + 2a_{i,i}b_{i,i}\rho(J'_A)\rho(J'_B).
\end{aligned} \tag{27}
$$

Thus, from (27), we have

$$
\begin{aligned}
\rho(A \circ B) &\leqslant \max_{i \neq j} \frac{1}{2}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} + \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 + 4a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho^2(J'_A)\rho^2(J'_B)\right]^{\frac{1}{2}}\right\} \\
&\leqslant \max_{1 \leqslant i \leqslant n} \frac{1}{2}\left\{2a_{i,i}b_{i,i} + 2a_{i,i}b_{i,i}\rho(J'_A)\rho(J'_B)\right\} \\
&= \max_{1 \leqslant i \leqslant n}\left\{a_{i,i}b_{i,i} + a_{i,i}b_{i,i}\rho(J'_A)\rho(J'_B)\right\} \\
&= (1 + \rho(J'_A)\rho(J'_B)) \max_{1 \leqslant i \leqslant n} a_{i,i}b_{i,i}.
\end{aligned}
$$

Hence, the bound in (18) is sharper than the known bound $(1 + \rho(J'_A)\rho(J'_B)) \max_{i \neq j} a_{i,i}b_{i,i}$ in [4].

From Theorem 3 we can obtain the following corollary:

Corollary 3. *Let $A = (a_{i,j}), B = (b_{i,j})$ be two $n \times n$ nonnegative matrices. If $a_{i,i}b_{j,j} \neq 0$ for all i, then we have*

$$
\begin{aligned}
|det(A \circ B)| &\leqslant [\rho(A \circ B)]^n \\
&\leqslant \max_{i \neq j} \frac{1}{2^n}\left\{a_{i,i}b_{i,i} + a_{j,j}b_{j,j} + \left[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 + 4a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho(J'_A)\rho(J'_B)\right]^{\frac{1}{2}}\right\}^n \\
&\leqslant (1 + \rho(J'_A)\rho(J'_B))^n \max_{1 \leqslant i \leqslant n}(a_{i,i}b_{i,i})^n.
\end{aligned}
$$

Example 3. Let

$$
A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 1 & 4 & 0.5 & 0.5 \\ 1 & 0 & 3 & 0.5 \\ 0.5 & 1 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0.5 & 0.5 & 0.5 \\ 1 & 1 & 1 & 1 \\ 0.5 & 0 & 2 & 0.5 \\ 0 & 1 & 1 & 2 \end{bmatrix}.
$$

Then

$$
A \circ B = \begin{bmatrix} 4 & 0 & 0.5 & 0.5 \\ 1 & 4 & 0.5 & 0.5 \\ 0.5 & 0 & 6 & 0.25 \\ 0 & 1 & 1 & 4 \end{bmatrix}.
$$

It is clear that $\rho(J'_A) = 0.8182$, $\rho(J'_B) = 1.1258$ and $\rho(A \circ B) = 6.3365$. By Theorem 6 in [4], we have

$$
\rho(A \circ B) \leqslant (1 + \rho(J'_A)\rho(J'_B)) \max_{1 \leqslant i \leqslant 4} a_{i,i}b_{i,i} = 11.5266.
$$

By Theorem 3 in this paper, we have

$$
\begin{aligned}
\rho(A \circ B) \leqslant \max_{i \neq j} \frac{1}{2}\Big\{&a_{i,i}b_{i,i} + a_{j,j}b_{j,j} + \Big[(a_{i,i}b_{i,i} - a_{j,j}b_{j,j})^2 \\
&+ 4a_{i,i}b_{i,i}a_{j,j}b_{j,j}\rho^2(J'_A)\rho^2(J'_B)\Big]^{\frac{1}{2}}\Big\} = 9.6221.
\end{aligned}
$$

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